

Knot Topology of Vacuum Space-Time and Vacuum Decomposition of Einstein's Theory

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Viewing Einstein's theory as the gauge theory of Lorentz group, we construct the most general vacuum connections which have vanishing curvature tensor and show that the vacuum space-time can be classified by the knot topology $\pi_3(S^3) \simeq \pi_3(S^2)$ of $\pi_3(SO(3,1))$. With this we obtain the gauge independent vacuum decomposition of Einstein's theory to the vacuum and gauge covariant physical parts. We discuss the physical implications of our result in quantum gravity.

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An important goal in theoretical physics is to construct a decent quantum gravity. For this purpose, we must understand the structure of the classical vacuum space-time first. To do that it is not enough for us to solve the vacuum Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \quad (1)$$

We must solve the true vacuum equation to obtain the most general vacuum space-time

$$R_{\mu\nu\rho\sigma} = 0. \quad (2)$$

The purpose of this Letter is to construct the most general vacuum connection in terms of the tetrad, and to obtain the generally invariant vacuum decomposition of Einstein's theory to the vacuum part and the generally covariant physical part. Our result shows that the vacuum space-time can be classified by the knot topology $\pi_3(S^3) \simeq \pi_3(S^2)$.

It is well known that Einstein's theory can be viewed as a gauge theory [1–3]. In particular it can be viewed as a gauge theory of Lorentz group in which the gravitational connection and curvature tensor become the gauge potential and field strength of Lorentz group [4]. In this view the gauge invariance and the general invariance become synonymous, because each gauge transformation has the corresponding general coordinate transformation. Adopting this view and imposing the vacuum isometry, we construct all possible vacuum connections in Einstein's theory, and show that they are identical to the vacuum solutions of $SU(2)$ gauge theory.

This immediately tells that the vacuum in R^4 space-time has the knot topology identical to the $SU(2)$ vac-

uum. This raises the possibility of vacuum tunneling in Einstein's theory. Of course, “the gravitational instantons” in Euclidian space-time classified by Euler-Poincare characteristics have been discussed before [5, 6]. But they are discussed without any knowledge of topological classification of the vacuum in real space-time. Obviously we have to know the topological structure of the vacuum first to discuss the tunneling. Our result shows that it is the spin structure (i.e., the tetrad) of the flat space-time which describes the knot topology of the vacuum.

Moreover, this allows the gauge invariant vacuum decomposition of the connection to the vacuum and gauge covariant physical parts. An important problem in Einstein's theory is to define the generally invariant momentum of gravitating particles which does not include the graviton. This has been thought to be impossible, because the covariant derivative always includes the connection. The vacuum decomposition not only makes this possible, but also allows us to obtain the vacuum decomposition of Einstein's theory itself.

To discuss the structure of vacuum space-time we consider the $SU(2)$ gauge theory first. Let \hat{n} be an arbitrary gauge covariant unit triplet which selects the Abelian direction. Imposing the isometry

$$D_\mu \hat{n} = (\partial_\mu + g \vec{A}_\mu \times) \hat{n} = 0, \quad (\hat{n}^2 = 1) \quad (3)$$

we have the Abelian projection of \vec{A}_μ [7, 8]

$$\vec{A}_\mu \rightarrow \hat{A}_\mu = A_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n}. \quad (A_\mu = \hat{n} \cdot \vec{A}_\mu) \quad (4)$$

An important feature of the “Abelian” binding potential \hat{A}_μ is that it retains the full $SU(2)$ gauge degrees of freedom, and is closed (transforms among itself) under the gauge transformation. Moreover, (4) allows us to have the Abelian decomposition [7, 8]

$$\vec{A}_\mu = \hat{A}_\mu + \vec{X}_\mu, \quad (\hat{n} \cdot \vec{X}_\mu = 0) \quad (5)$$

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where \vec{X}_μ is the gauge covariant valence potential. Notice that the decomposition is gauge independent. Once the Abelian direction is chosen the decomposition follows automatically, independent of the choice of a gauge.

Now, let \hat{n}_i ($i = 1, 2, 3$) with $\hat{n}_3 = \hat{n}$ be an orthonormal right-handed basis, and impose the vacuum isometry which assures the vanishing field strength

$$\forall_i \quad D_\mu \hat{n}_i = 0. \quad (6)$$

Solving (6) we obtain the most general vacuum potential $\hat{\Omega}_\mu$ [9]

$$\begin{aligned} \hat{\Omega}_\mu &= C_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n} = C_\mu^k \hat{n}_k, \\ C_\mu^k &= \frac{1}{2g} \epsilon_{ij}^k (\hat{n}_i \cdot \partial_\mu \hat{n}_j), \quad C_\mu = -C_\mu^3. \end{aligned} \quad (7)$$

Notice that, although the vacuum is fixed by three isometries, it is essentially fixed by \hat{n} . This is because \hat{n}_1 and \hat{n}_2 are uniquely determined by \hat{n} , up to a $U(1)$ gauge transformation which leaves \hat{n} invariant. With

$$\hat{n} = \begin{pmatrix} \sin \alpha \cos \beta \\ \sin \alpha \sin \beta \\ \cos \alpha \end{pmatrix}, \quad (8)$$

we have [9]

$$\begin{aligned} C_\mu^1 &= \frac{1}{g} (\sin \gamma \partial_\mu \alpha - \sin \alpha \cos \gamma \partial_\mu \beta), \\ C_\mu^2 &= \frac{1}{g} (\cos \gamma \partial_\mu \alpha + \sin \alpha \sin \gamma \partial_\mu \beta), \\ C_\mu^3 &= \frac{1}{g} (\cos \alpha \partial_\mu \beta + \partial_\mu \gamma), \end{aligned} \quad (9)$$

where the angle γ represents the $U(1)$ angle which leaves \hat{n} invariant.

It is well known that the $SU(2)$ vacuum has the $\pi_3(S^3)$ topology [10, 11]. This topology is naturally inscribed in our vacuum (7). The vacuum quantum number which determines the wrapping number of the mapping from the compactified space S^3 to the $SU(2)$ space S^3 is given by the Chern-Simon index of $\hat{\Omega}_\mu$ [9, 10]

$$n = -\frac{g^3}{96\pi^2} \int \epsilon_{0\alpha\beta\gamma} \epsilon_{ijk} C_\alpha^i C_\beta^j C_\gamma^k d^3x. \quad (10)$$

But with the Hopf fibering this $\pi_3(S^3)$ can also be interpreted as $\pi_3(S^2)$ of the mapping from S^3 to the coset space $SU(2)/U(1)$ defined by \hat{n} [9]. Of course, the topologically distinct vacua become unstable under the quantum fluctuation because of the vacuum tunneling made by instantons. This leads us to introduce the θ -vacuum

$$|\theta\rangle = \sum_n \exp(in\theta) |n\rangle, \quad (11)$$

in non-Abelian gauge theory [10, 11].

Another important application of (7) is that it allows us to have the vacuum decomposition

$$\vec{A}_\mu = \hat{\Omega}_\mu + \vec{Z}_\mu, \quad \vec{Z}_\mu = (A_\mu + C_\mu) \hat{n} + \vec{X}_\mu. \quad (12)$$

Here again $\hat{\Omega}_\mu$ (just like \hat{A}_μ) has the full gauge degrees of freedom, and \vec{Z}_μ transforms covariantly. Moreover, the decomposition is gauge independent.

The vacuum decomposition allows us to define the gauge invariant canonical momentum of charged particles which does not include the gauge boson, which has been thought to be impossible [12]. Indeed, in QCD (12) allows us to define the gauge invariant canonical momentum of quarks which does not include the gluons. This has important applications. In particular, this makes a gauge independent decomposition of nucleon momentum and spin to those of quarks and gluons possible [13].

Now, treating Einstein's theory as a gauge theory of Lorentz group, we can find the most general vacuum connection imposing the vacuum isomtry. Let J^{ab} ($a, b = 0, 1, 2, 3$) be the six generators of Lorentz group which can be expressed by the rotation and boost generators L_i and K_i ($i = 1, 2, 3$). Let \mathbf{p} (or p^{ab}) be an adjoint representation of Lorentz group and $\tilde{\mathbf{p}}$ (or $\tilde{p}^{ab} = \epsilon_{abcd} p^{cd}/2$) be its dual partner, and let \vec{m} and \vec{e} be the magnetic and electric components of \mathbf{p} which correspond to 3-dimensional rotation and boost,

$$\mathbf{p} = \begin{pmatrix} \vec{m} \\ \vec{e} \end{pmatrix}, \quad \tilde{\mathbf{p}} = \begin{pmatrix} \vec{e} \\ -\vec{m} \end{pmatrix}, \quad \tilde{\tilde{\mathbf{p}}} = -\mathbf{p}. \quad (13)$$

To proceed further let $\mathbf{\Gamma}_\mu$ (or Γ_μ^{ab}) be the gauge potential of Lorentz group which describes the spin connection, and $\mathbf{R}_{\mu\nu}$ (or $R_{\mu\nu}^{ab}$) be the curvature tensor

$$\mathbf{R}_{\mu\nu} = \partial_\mu \mathbf{\Gamma}_\nu - \partial_\nu \mathbf{\Gamma}_\mu + \mathbf{\Gamma}_\mu \times \mathbf{\Gamma}_\nu. \quad (14)$$

Now, consider the following isometry

$$D_\mu \mathbf{p} = (\partial_\mu + \mathbf{\Gamma}_\mu \times) \mathbf{p} = 0. \quad (15)$$

This automatically assures

$$D_\mu \tilde{\mathbf{p}} = (\partial_\mu + \mathbf{\Gamma}_\mu \times) \tilde{\mathbf{p}} = 0, \quad (16)$$

which tells that the isometry in Lorentz group always includes the dual partner. To verify this we decompose the gauge potential $\mathbf{\Gamma}_\mu$ into the 3-dimensional rotation and boost parts \vec{A}_μ and \vec{B}_μ ,

$$\mathbf{\Gamma}_\mu = \begin{pmatrix} \vec{A}_\mu \\ \vec{B}_\mu \end{pmatrix}. \quad (17)$$

With this both (15) and (16) can be written as [14]

$$\begin{aligned} D_\mu \vec{m} &= \vec{B}_\mu \times \vec{e}, \quad D_\mu \vec{e} = -\vec{B}_\mu \times \vec{m}, \\ D_\mu &= \partial_\mu + \vec{A}_\mu \times. \end{aligned} \quad (18)$$

This confirms that (15) and (16) are actually identical to each other, which tells that the isometry in Lorentz group always comes in pairs.

Let \mathbf{l}_i and \mathbf{k}_i be an orthonormal basis of the adjoint representation of Lorentz group which describe the rotation and boost. And let \hat{n}_i be the orthonormal triplets of the $SU(2)$ subgroup of Lorentz group,

$$\mathbf{l}_i = \begin{pmatrix} \hat{n}_i \\ 0 \end{pmatrix}, \quad \mathbf{k}_i = \begin{pmatrix} 0 \\ \hat{n}_i \end{pmatrix} = -\tilde{\mathbf{l}}_i. \quad (19)$$

Now, it must be clear that the most general vacuum connection which guarantees $\mathbf{R}_{\mu\nu} = 0$ is described by the following vacuum isometry

$$\forall_i \quad D_\mu \mathbf{l}_i = 0, \quad D_\mu \mathbf{k}_i = -D_\mu \tilde{\mathbf{l}}_i = 0. \quad (20)$$

Actually the vacuum needs only one of them, because they are dual to each other.

To obtain the most general vacuum connection we have to solve (20). To do that, notice that in 3-dimensional notation (20) is written as

$$\forall_i \quad D_\mu \hat{n}_i = \vec{B}_\mu \times \hat{n}_i, \quad D_\mu \hat{n}_i = -\vec{B}_\mu \times \hat{n}_i. \quad (21)$$

This has the unique solution

$$\vec{A}_\mu = \hat{\Omega}_\mu, \quad \vec{B}_\mu = 0. \quad (22)$$

where $\hat{\Omega}_\mu$ is precisely the vacuum potential (7) of the $SU(2)$ subgroup. So we have the most general vacuum connection Ω_μ which yields vanishing curvature tensor

$$\Gamma_\mu = \Omega_\mu = \begin{pmatrix} \hat{\Omega}_\mu \\ 0 \end{pmatrix}. \quad (23)$$

This shows that the vacuum connection of Einstein's theory is given by the vacuum potential of $SU(2)$ gauge theory.

Obviously this tells that the vacuum space-time has exactly the same structure as the vacuum of $SU(2)$ gauge theory. In particular, this tells that the vacuum (in R^4 space-time) can be classified by the knot topology $\pi_3(S^3) \simeq \pi_3(S^2)$. This may not be surprising, considering the fact $\pi_3(SO(3,1)) \simeq \pi_3(SU(2))$.

At this point, one may wonder if one could describe this vacuum topology in terms of the metric. To answer this, consider a flat space-time which has an R^4 topology. Choosing a proper coordinate basis ∂_μ , we can always transform the flat metric and torsionless flat Levi-Civita connection to acquire the trivial form,

$$g_{\mu\nu} = \eta_{\mu\nu}, \quad \Gamma_{\mu\nu}^\alpha = 0. \quad (24)$$

Now, introduce an orthonormal (i.e., tetrad) basis e_a by

$$\begin{aligned} e_a &= \hat{n}_a^\alpha \partial_\alpha, \quad \partial_\mu = \hat{n}_\mu^a e_a, \quad (\hat{n}_\mu^a \hat{n}_{a\nu} = \eta_{\mu\nu}) \\ e_0 &= \partial_0, \quad e_i = \hat{n}_i^\alpha \partial_\alpha, \quad (\hat{n}_0^\alpha = \delta_0^\alpha, \quad \hat{n}_i^0 = 0) \\ [e_a, e_b] &= f_{ab}^c e_c, \\ f_{ab}^c &= (\hat{n}_a^\mu \partial_\mu \hat{n}_b^\nu - \hat{n}_b^\mu \partial_\mu \hat{n}_a^\nu) \hat{n}_\nu^c. \end{aligned} \quad (25)$$

Using the identity

$$\mathcal{D}_\mu e_\nu^a \equiv \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\alpha e_\alpha^a + \Gamma_{\mu b}^a e_\nu^b = 0, \quad (26)$$

we can easily show that the flat connection (24) acquires the following form in the tetrad basis

$$\Gamma_\mu^{ab} = \frac{\eta^{\alpha\beta}}{2} (\hat{n}_a^\alpha \partial_\mu \hat{n}_b^\beta - \hat{n}_b^\alpha \partial_\mu \hat{n}_a^\beta). \quad (27)$$

This is precisely the vacuum connection (23). Moreover (with $\hat{n}_\mu^a \hat{n}_{a\nu} = \eta_{\mu\nu}$) we can show that this is nothing but the flat connection (24) expressed in the tetrad basis,

$$\Gamma_\mu^{ab} = \frac{1}{2} \hat{n}_\mu^c (f_c^{ab} - f_c^{ba} - f_c^{ab}) = \Omega_\mu^{ab}. \quad (28)$$

This tells that it is the spin structure of the flat space-time, the non-trivial configuration of tetrad, which describes the vacuum topology. In particular, this shows that metric and torsion have no role in the vacuum topology.

It has generally been assumed that the tetrad is no more fundamental than the metric. Moreover, ever since Feynman tried to quantize the gravity with the metric, the metric has been treated as the quantum field of gravity [15]. But our result shows that the tetrad is more fundamental. In fact, there is an unshakable evidence for this: The graviton which couples to spinors in Feynman diagrams is the tetrad, not the metric. So, when spinors are present, we have no choice but to treat the tetrad as the fundamental field of gravity.

Clearly our result raises the possibility of the vacuum tunneling in Einstein's theory. Candidates of the "gravitational instanton" which have finite Euclidian action have been discussed by many authors [5, 6]. But they have been proposed without any reference to the above vacuum topology. Whether any of these, or any unknown gravito-instantons, can actually demonstrate the tunneling is an interesting question worth further study [16].

Assuming the tunneling, we can certainly introduce the θ -vacuum in gravity. The question is whether this θ -vacuum could be the physical vacuum in quantum gravity. Of course, in gauge theory the answer is no. But in Einstein's theory the θ -vacuum could play an important role, because a gravito-instanton (if exists) could have a vanishing action and thus have the maximum tunneling probability. This could make the θ -vacuum much more important in quantum gravity.

But perhaps a more important application of (23) is that it provides us the Lorentz invariant vacuum decomposition of an arbitrary connection to the vacuum and gauge covariant physical parts by

$$\begin{aligned} \Gamma_\mu &= \Omega_\mu + \mathbf{Z}_\mu, \\ \mathbf{R}_{\mu\nu} &= \bar{D}_\mu \mathbf{Z}_\nu - \bar{D}_\nu \mathbf{Z}_\mu + \mathbf{Z}_\mu \times \mathbf{Z}_\nu, \end{aligned} \quad (29)$$

where $\bar{D}_\mu = \partial_\mu + \Omega_\mu \times$. To understand the meaning of this, let α be an infinitesimal gauge parameter. Now,

under the gauge transformation

$$\delta \mathbf{T}_\mu = D_\mu \boldsymbol{\alpha}, \quad \delta \mathbf{l}_i = -\boldsymbol{\alpha} \times \mathbf{l}_i, \quad \delta \mathbf{k}_i = -\boldsymbol{\alpha} \times \mathbf{k}_i,$$

we have

$$\delta \boldsymbol{\Omega}_\mu = \bar{D}_\mu \boldsymbol{\alpha}, \quad \delta \mathbf{Z}_\mu = -\boldsymbol{\alpha} \times \mathbf{Z}_\mu. \quad (30)$$

This tells that $\boldsymbol{\Omega}_\mu$ retains the full Lorentz gauge degrees of freedom and forms a closed connection space by itself. Moreover, \mathbf{Z}_μ transforms covariantly and thus can be interpreted to represent the physical part of the connection. Most importantly, the decomposition is gauge independent. Once \mathbf{l}_i and \mathbf{k}_i are chosen, the decomposition follows independent of the gauge.

Certainly we can have a similar generally invariant vacuum decomposition in the coordinate basis. Using (26) we can transform $\boldsymbol{\Omega}_\mu$ and \mathbf{Z}_μ to the coordinate basis,

$$\begin{aligned} \Omega_{\mu\nu}^\alpha &= \Omega_\mu^{ab} e_{a\nu} e_b^\alpha + e_a^\alpha \partial_\mu e_\nu^a, \\ Z_{\mu\nu}^\alpha &= Z_\mu^{ab} e_{a\nu} e_b^\alpha. \end{aligned} \quad (31)$$

Notice that $\Omega_{\mu\nu}^\alpha$ (just like Ω_μ^{ab}) transforms exactly as a connection under the general coordinate transformation, and forms its own closed connection space. In contrast, $Z_{\mu\nu}^\alpha$ (just like Z_μ^{ab}) transforms covariantly. With this we have the vacuum decomposition of an arbitrary connection to the vacuum part and the generally covariant physical part in the coordinate basis [17]

$$\begin{aligned} \Gamma_{\mu\nu}^\alpha &= \Omega_{\mu\nu}^\alpha + Z_{\mu\nu}^\alpha, \\ R_{\mu\nu\rho}^\sigma &= \bar{\nabla}_\mu Z_{\nu\rho}^\sigma - \bar{\nabla}_\nu Z_{\mu\rho}^\sigma + Z_{\mu\alpha}^\sigma Z_{\nu\rho}^\alpha \\ &\quad - Z_{\nu\alpha}^\sigma Z_{\mu\rho}^\alpha, \end{aligned} \quad (32)$$

where $\bar{\nabla}_\mu$ is the generally covariant derivative made of the vacuum connection. Clearly the decomposition is

independent of choice of the general coordinates, because it holds in any coordinate basis.

A fundamental problem in gauge theory is to obtain a gauge independent canonical momentum of a charged particle which does not include the gauge boson [12]. The vacuum decomposition (12) makes this possible [13]. Our vacuum decomposition in Einstein's theory plays an equally important role. To see this consider a gravitational binary system made of two particles, and try to decompose the momentum and spin of this system to those of the constituent particles and pure gravity in a generally invariant way. Without the vacuum decomposition this is impossible. But now this becomes possible, because (32) allows us to define the generally invariant canonical momentum of the particles which does not include the graviton [17].

In general the vacuum decomposition allows us to have the generally invariant energy-momentum and angular momentum decompositions of a composite system to those of pure gravitational and non-gravitational parts. More importantly, this provides us the generally invariant vacuum decomposition of Einstein's theory itself, and allows us to reformulate the theory essentially in terms of the generally covariant physical quantities. This will have far reaching consequences. In particular, this could play an important role in quantum gravity.

The detailed discussions on these and related subjects will be presented in a separate paper [14, 17].

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- [1] R. Utiyama, Phys. Rev. **101**, 1597 (1956); K. Hayashi and T. Nakano, Prog. Theor. Phys. **38**, 491 (1967).
 - [2] T. W. B. Kibble, J. Math. Phys. **2**, 212 (1961); D. Sciama, Rev. Mod. Phys. **36**, 463 (1964).
 - [3] Y. M. Cho, Phys. Rev. **D14**, 2521 (1976).
 - [4] Y. M. Cho, Phys. Rev. **D14**, 3335 (1976). See also, F. Hehl, P. Heyde, G. Kerlick, and J. Nester, Rev. Mod. Phys. **48**, 393 (1976).
 - [5] S. Hawking, Phys. Lett. **A60**, 81 (1977); D. Page, Phys. Lett. **B78**, 239 (1978); G. Gibbons and S. Hawking, Phys. Lett. **B78**, 430 (1978).
 - [6] See also, T. Eguchi, P. Gilkey, and A. Hanson, Phys. Rep. **66**, 213 (1980), and references therein.
 - [7] Y. M. Cho, Phys. Rev. **D21**, 1080 (1980); Phys. Rev. **D62**, 074009 (2000).
 - [8] Y. M. Cho, Phys. Rev. Lett. **46**, 302 (1981); Phys. Rev. **D23**, 2415 (1981); W. S. Bae, Y. M. Cho, and S. W. Kimm, Phys. Rev. **D65**, 025005 (2001).
 - [9] Y. M. Cho, Phys. Lett. **B644**, 208 (2006).
 - [10] G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976).
 - [11] A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin, Phys. Lett. **B59**, 85 (1975); Y. M. Cho, Phys. Lett. **B81**, 25 (1979).
 - [12] J. Jauch and F. Rohrlich, *The Theory of Photons and Electrons*, Springer-Verlag (Berlin) 1976; R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics Vol. III* (Addison-Wesley) 1965.
 - [13] Y. M. Cho, M. L. Ge, and P. M. Zhang, nucl-th/1010.1080; nucl-th/1102.1130, submitted to Phys. Rev. **C**.
 - [14] Y. M. Cho, S. H. Oh, Sang-Woo Kim, gr-qc/1102.3490, submitted to Phys. Rev. **D**.
 - [15] R. Feynman, Acta Phys. Pol. **24**, 697 (1967); B. DeWitt, Phys. Rev. **162**, 1195 (1967); 1239 (1967).
 - [16] Y. M. Cho and D. G. Pak, Class. Quant. Grav. **28**, 155008 (2011).
 - [17] Y. M. Cho and S. H. Oh, to be published.